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RESONANCES AND CHAOS IN PARAMETRIC SYSTEMS†

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Time-periodic systems close to two-dimensional non-linear Hamiltonian systems are studied in the case when the perturbation contains non-linear parametric terms and is non-conservative. A condition for the existence of new regimes in the resonance zones, namely, regular two-frequency and irregular "quasi-attractors" is established. The problem of the transition from the resonance case to the non-resonance one as the difference in tuning frequency is altered is solved by analysing self-excited oscillatory truncated systems determining the topology of the resonance zones. Using a specific example, it is shown that the numerical and analytic results are in good agreement in the quasi-conservative case.

1. FORMULATION OF THE PROBLEM

We consider systems of the form

$$dx/dt = \partial H(x,y)/\partial y + \varepsilon g(x,y,\nu t), \quad dy/dt = -\partial H(x,y)/dx + \varepsilon f(x,y,\nu t) \tag{1.1}$$

where ε and ν are parameters ($|\varepsilon|$ is small), and g and f are periodic functions with respect to $\varphi = \nu t$ with period 2π.

We shall assume that the unperturbed Hamiltonian system

$$dx/dt = \partial H(x,y)/\partial y, \quad dy/dt = -\partial H(x,y)/dx \tag{1.2}$$

is non-linear and has at least one cell D filled with closed phase curves. The Hamiltonian H as well as g and f will be assumed to be sufficiently smooth (analytic) in x, y in a domain $G \subset R^2$ or $G \subset R^1 \otimes S^1$. We shall also assume that g and f are continuous with respect to t.

We assume that the following condition is satisfied.

Condition A. $\partial g/\partial x + \partial f/\partial y \not\equiv 0$.

The condition implies that (1.1) is a non-conservative system.

Along with (1.1), we shall consider the autonomous system

$$dx/dt = \partial H(x,y)/\partial y + \varepsilon \bar{g}(x,y), \quad dy/dt = -\partial H(x,y)/dx + \varepsilon \bar{f}(x,y) \tag{1.3}$$

where $\bar{g} = \langle g \rangle_\varphi$ and $\bar{f} = \langle f \rangle_\varphi$.

We assume that the following condition is satisfied.

Condition B. System (1.3) has a finite set of rough limit cycles in cell D.

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The purpose of this paper is to solve the important problem of new regimes arising in the resonance zones. Both regular and irregular regimes ("quasi-attractors") are possible. (In the problem under consideration the term "quasi-attractor" [1] means that there is a homoclinic Poincaré structure with an attracting neighbourhood.)

The properties of the quasi-attractor itself will not be analysed numerically here, as there are now many papers devoted to this subject (for example, [2]).

Moreover, in this paper we use system (1.1)

$$dx/dt = y, \quad dy/dt = -x - x^3 + (P_1 + P_2x^2 + P_3x\sin(\nu t))y + P_4\sin(\nu t) \tag{1.4}$$

where P_i are parameters, as a specific example to illustrate the theoretical results obtained for small $|\varepsilon|$ and establish that the theory is in good agreement with experiments for $|\varepsilon|$ other than small ones. It is required only that the non-conservative terms should be small, i.e. that the parameters P_1, P_2, P_3 in (1.4) should be small. New regimes exist in the resonance zones due to the presence of non-linear parametric terms in f and g (in (1.4) it is the term $P_3xy\sin(\nu t)$). The existence of such regimes in systems of the form (1.1) was first pointed out in [3, 4]. The present paper, in which the theory of essentially non-linear self-oscillatory systems is developed, is closely related to [4, 5].

When studying systems (1.1) and (1.3) it proves convenient to change from the variables x, y to the action (I) and angle (θ). In terms of the new variables (1.1) takes the form

$$\begin{gathered} dI/dt = \varepsilon F(I,\theta,\varphi), \quad d\theta/dt = \omega(I) + \varepsilon R(I,\theta,\varphi), \quad d\varphi/dt = \nu \\ (F = gX_\theta - fY_\theta, \quad R = -gX_I + fY_I) \end{gathered} \tag{1.5}$$

where $\omega(I)$ is the characteristic oscillation frequency, which can be determined from (1.2).

System (1.5) is defined on the direct product $\delta \otimes S^1 \otimes S^1 = \delta \otimes T^2$, where δ is the image of the interval $[h\min, h\max]$ ($[h \subset [h\min, h\max]$, where $H(x, y) = h$ is the "energy" integral for (1.2)). Since ω and ν are commensurable

$$\omega(I) = q\nu/p, \quad q, p \subset N \tag{1.6}$$

In other words, because of the resonances, the completely averaged system

$$dI/dt = \varepsilon B(I), \quad B(I) = \langle F \rangle_{\varphi,\theta} \tag{1.7}$$

does not, in general, determine the behaviour of the solutions of (1.5) in the global domain corresponding to the variation of I in the interval δ.

The equation

$$\mathrm{B}(I) = 0 \tag{1.8}$$

is called the Poincaré–Pontryagin generating equation. Its simple real roots determine the structurally stable limit cycles in system (1.3). Condition B is equivalent to the following condition: the set of simple roots of Eq. (1.8) on δ is bounded. Therefore, if $\mathrm{B}(I_0) = 0$, then $\mathrm{dB}(I_0)\mathrm{d}I \neq 0$. This makes it possible [5] to establish the qualitative behaviour of solutions in the global domain $\delta \otimes T^2$. It suffices to establish the behaviour of solutions in the neighbourhoods of the bounded set of resonance levels $I = I_{pq}(H(x, y) = h_{pq})$, where I_{pq} can be determined from (1.6) and $I_{pq} \subset \delta$.

2. THE STRUCTURE OF RESONANCE ZONES IN PARAMETRIC SYSTEMS

The behaviour of solutions in the neighbourhoods

$$U=\{(I,\theta): I_{pq}-C\mu<I<I_{pq}+C\mu,\quad 0\le\theta\le 2\pi,\quad C=\text{const}\},\ \mu=\sqrt{\varepsilon}$$

of the induced resonance levels $I=I_{pq}(H(x,\ y)=h_{pq})$ can be determined apart from terms $O(\mu^2)$ [3, 4] from the pendulum-type equation

$$\frac{d^2\nu}{d\tau^2}-bA(\nu;I_{pq})=\mu\sigma(\nu;I_{pq})\frac{d\nu}{d\tau} \tag{2.1}$$

$$b=d\omega(I_{pq})/dI,\ \tau=\mu t,\ A(\nu;I_{pq})=\frac{1}{2\pi p}\int_0^{2\pi p}F(I_{pq},\nu+q\varphi/p,\varphi)d\varphi$$

$$\sigma(\nu;I_{pq})=\frac{1}{2\pi p}\int_0^{2\pi p}\left[\frac{dF(I_{pq},\nu+q\varphi/p,\varphi)}{\partial I}+\frac{\partial R(I_{pq},\nu+q\varphi/p,\varphi)}{\partial\nu}\right]d\varphi$$

The smallest period of $A(\nu,\ I_{pq})$ and $\sigma(\nu,\ I_{pq})$ is seen to be $2\pi/p$ [3, 6]. The following relationship holds (stated without proof in [4])

$$\sigma(v,I_{pq})=\frac{1}{2\pi p}\int_0^{2\pi p}\left(\frac{\partial g(X,Y,\varphi)}{\partial X}+\frac{\partial f(X,Y,\varphi)}{\partial Y}\right)d\varphi \tag{2.2}$$

where $X=X(I_{pq},\ \nu+q\varphi/p)$, $Y=Y(I_{pq},\ \nu+q\varphi/p)$ is the unperturbed solution at the level $I=I_{pq}$.

Indeed, the expressions for F and R in (1.5) imply that

$$\partial F(I,\theta,\varphi)/\partial I+\partial R(I,\theta,\varphi)/\partial\theta=(\partial g/\partial x+\partial f/\partial y)(x_\theta y_I-x_I y_\theta)$$

Since $(x,\ y)\Rightarrow(I,\ \theta)$ is a canonical transformation, $(x_\theta y_I-x_I y_\theta)=1$. Hence we obtain (2.2).

Relationship (2.2) implies the following result.

Theorem 1. If the divergence of the vector field in (2.1) depends on V, then the divergence of the vector field in the original system (1.1) contains terms depending on the time t and on one or both coordinates.

In many cases the converse assertion is true (for example, for (1.4)).

The terms mentioned in Theorem 1 are called non-linear parametric terms.

We will now investigate Eq. (2.1). First, we note that

$$A(\nu;I_{pq})=A_*(\nu;I_{pq})+B(I_{pq}),\quad B=\langle A\rangle_\nu. \tag{2.3}$$

$$\sigma(\nu;I_{pq})=\sigma_*(\nu;I_{pq})+B_1(I_{pq}),\quad B_1=\langle\sigma\rangle_\nu$$

where $B(I)$ is the generating function of the autonomous system (1.3) and B_1 is the derivative of $B(I)$ The case when $\sigma(\nu;\ I_{pq})$ has constant sign has been studied fairly fully (for example, [3–7]). We shall therefore consider the case when σ has alternating sign. In other words, we shall study the new structure of resonance zones due to the non-linear parametric terms in the perturbations. It follows from (2.3) that in this case

$$|B_1(I_{pq})|<\max_\nu|\sigma_*(\nu,I_{pq})| \tag{2.4}$$

When studying the pendulum equation (2.1) we shall distinguish between the cases when (a)

$B(I_{pq}) \neq 0$ and (b) $B(I_{pq}) = 0$. In case (b) system (3) has a structurally stable limit cycle in a neighbourhood of the level $H(x, y) = h_{pq}$. There is no such cycle in case (a).

Case (*a*). Neglecting terms of order μ in (2.1), we arrive at the integrable equation

$$d^2v / d\tau^2 - bA(v, I_{pq}) = 0 \tag{2.5}$$

If

$$|B(I_{pq})| > \max_{v} |A_*(v, I_{pq})|$$

then (2.5) has no equilibrium states. The resonance level $I = I_{pq}$ is then referred to as "passable" [3, 4]. Note that this term has been derived from the topology of the resonance zone, as opposed to the corresponding term used in physics, where it means a change in the perturbation frequency ν. In the case under consideration there are no periodic solutions in the vicinity of the resonance level.

The most interesting case is when Eq. (2.5) has equilibrium states, i.e. when the condition

$$|B(I_{pq})| < \max_{v} |A_*(v, I_{pq})| \tag{2.6}$$

is satisfied. In this case the resonance level $I = I_{pq}$ is said to be partially passable (for details see [4, 6]).

Equation (2.1) is equivalent to an autonomous system, being a special case of (1.3). Under condition (2.4), there may be limit cycles in this system. To determine the latter one must construct the Poincaré–Pontryagin generating function.

In Fig. 1(a) we show the phase picture of Eq. (2.1) under conditions (2.4) and (2.6). There is a unique limit cycle. If the cycle lies outside a neighbourhood of the separatrix loop of Eq. (2.5), there is a corresponding two-dimensional invariant torus in the original system. Since the period of the limit cycle of Eq. (2.1) is of order 1/μ in t, in this case we have long-time pulsation regimes in the original system (1.1) (the generatrices of the torus have different order).

However, if the limit cycle lies in a neighbourhood of the separatrix loop (whose width in terms of I is known to be of order $\exp(-1/\mu)$), then the two-dimensional invariant torus in the original system (1.1) is destroyed. A bifurcation setting in which the cycle is caught inside the separatrix loop is shown in Fig. 1(b). The non-autonomous terms, which were discarded when deriving (1.2), lead to he homoclinic structure shown in Fig. 1(c) for the Poincaré map with $p = 3$. Because of the presence of non-compact separatrices, in this case one can talk of an irregular transfer process only.

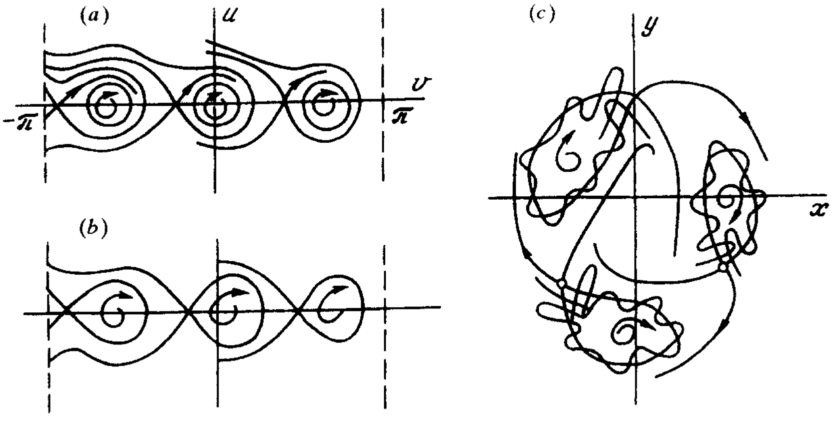

Fig. 1.

Case (*b*). In this case Eq. (2.1) always possesses equilibrium states and we have a third kind of resonance zone, namely, a non-passable resonance zone. To better understand the structure of such a zone, we introduce in (2.1) the difference in frequency γ between the level $I=I_{pq}$ and the level $I=I_0$, in the neighbourhood of which the autonomous system (1.3) has a limit cycle

$$B(I_{pq})=(dB(I_0)/dI)(I_{pq}-I_0)+O((I_{pq}-I_0)^2)\simeq\gamma\mu \tag{2.7}$$

Then Eq. (2.1) can be rewritten as the system

$$du/d\tau=A_*(v;I_{pq})+\mu(\sigma(v;I_{pq})u+\gamma),\quad dv/d\tau=bu \tag{2.8}$$

For non-degenerate resonance zones, which are considered here, $b\neq 0$.

In (2.8) we will change the variables (v, u) to the action J and angle L (both in the oscillatory and the rotational zones) and average the resulting system over the "fast" angular variable L. As a result, we arrive at the equation $dJ/d\tau=\mu b\Phi(J)/2\pi$, where

$$\Phi(J)=\begin{cases}\dfrac{1}{\Omega}\displaystyle\int_0^{2\pi}\sigma(v(J,L);I_{pq})u^2(J,L)dL & \text{in the oscillatory zone}\\ \dfrac{1}{\Omega}\displaystyle\int_0^{2\pi}\sigma(v(J,L);I_{pq})u^2(J,L)dL\pm\dfrac{2\pi\gamma}{b} & \text{in the rotational zone}\end{cases}$$

Here Ω is the characteristic frequency and $0<J<J_c$, where J_c corresponds to the contour in the "unperturbed" system

$$du/d\tau=A_*(v;I_{pq}),\quad dv/d\tau=bu \tag{2.9}$$

formed by the saddle and two separatrix loops enclosing the phase cylinder.

The function $\Phi(J)$ is discontinuous at the point $J=J_c$ when $\gamma\neq 0$. We shall therefore use Mel'nikov's formula [8] to determine the relative location of the separatrices, which in system (2.9) constitute the contour formed by the outer separatrix loops

$$\Delta=\mu\Delta_1^{\pm}+O(\mu^2)$$

$$\Delta_1^{\mp}=b\int_{-\infty}^{\infty}(\sigma_*(v_0;I_{pq})+B_1(I_{pq}))u_0^2\,d\tau\mp 2\pi\gamma$$

Here v_0, u_0 is a solution of (2.9) on the contour consisting of the saddle and the outer separatrix loops. Setting

$$d=\max_v|\sigma_*(v;I_{pq})|=\|\sigma_*\|,\quad a=|B_1(I_{pq})|/d$$

we find from the formula for $\Delta_1^{\pm}$ that $\Delta_1^{\pm}=d(\alpha+\beta a)\pm 2\pi\gamma$, where

$$\alpha=b\int_{-\infty}^{\infty}\bar{\sigma}(v_0;I_{pq})u_0^2d\tau,\quad \beta=b\int_{-\infty}^{\infty}u_0^2d\tau,\quad \bar{\sigma}=\frac{\sigma_*}{\|\sigma_*\|}$$

From the condition $\Delta_1^{\pm}=0$ we get

$$\gamma=\gamma^{\pm}=\mp d(\alpha+\beta a)/2\pi \tag{2.10}$$

In system (2.8) the upper contour exists when $\gamma=\gamma^+$, and the lower contour exists when $\gamma=\gamma^-$. Relationship (2.10) defines two straight lines in the (a, γ)-plane. They intersect each other at $(a_*, 0)$, where $a_* = -\alpha/\beta$. When $|a|>1$ the function $\sigma(\nu;\ I_{pq})$ has a constant sign, and when $|a|<1$ it has an alternating sign.

By virtue of (2.4), the second case is of interest since the first one has been considered before in [4, 5]. The case $|a|<1$ is special in that limit cycles can exist in system (2.8) both in the oscillatory and rotational domains, for which there are no generating cycles in system (1.3). The following bifurcations giving rise to limit cycles in (2.8) are possible: (a) a structurally unstable focus, (b) a separatrix loop (contour), and (c) a condensation of trajectories. However, if one is not interested in the number of limit cycles, it suffices to consider the case when there is no more than one limit cycle in the oscillatory domain (again, here one can add the words "apart from an even number"). Then one can state a general assertion on the variation of the qualitative behaviour of the trajectories of (2.8) as the difference in frequency tuning is altered, similarly as in [4]. However, prior to this we consider in detail the problem for the case when f and g are trigonometric polynomials of degree N in φ. Then, by (1.5) and (2.1), A and σ are also trigonometric polynomials of degree not exceeding $N_1 \leqslant N$

$$-bA_*(\nu;I_{pq}) = \sum_{i=1}^{N_1}(a_i\cos(ip\nu)+b_i\sin(ip\nu)),\quad \sigma_*(\nu;I_{pq}) = \sum_{i=1}^{N}(d_i\cos(ip\nu)+c_i\sin(ip\nu)) \tag{2.11}$$

It follows from (2.1) and (2.2) that, in general, different harmonics in the perturbation contribute to A and σ. This means that harmonics with different numbers can predominate in (2.11). We leave only these fundamental harmonics in (2.11) To fix our ideas suppose that the first harmonic predominates in the expression for $A_*(\nu;\ I_{pq})$ and the nth harmonic predominates in the expression for $\sigma_*(\nu;\ I_{pq})$. Then, by substituting $\gamma b \Rightarrow \gamma$, one can represent (2.8) in the form of the equation

$$d^2\nu/d\tau^2 + r\sin(p\nu+\psi) = \mu[(d_n\cos(np\nu)+c_n\sin(np\nu)+B_1)(d\nu/d\tau)+\gamma] \tag{2.12}$$

$$(r=\sqrt{a_1^2+b_1^2},\quad \psi=\operatorname{arctg}(b_1/a_1))$$

After the substitution

$$z=p\nu+\psi,\ \sqrt{rp}\tau\Rightarrow\tau,\ \mu/\sqrt{rp}\Rightarrow\mu,\ \gamma/\sqrt{p/r}\Rightarrow\gamma$$

$$\tilde d_n = d_n\cos(n\psi)-c_n\sin(n\psi),\ \tilde c_n = d_n\sin(n\psi)+c_n\cos(n\psi)$$

we arrive at the equation (the prime denotes a derivative with respect to τ)

$$z''+\sin(z)=\mu[(\tilde d_n\cos(nz)+\tilde c_n\sin(nz)+B_1)z'+\gamma]$$

The term $\mu\tilde c_n z'\sin(z)$ in this equation is conservative, i.e. it does not contribute to the generating function $\Phi(J)$. We shall therefore discard it. Next, setting $B_1/\tilde d_n = a,\ \gamma/\tilde d_n \Rightarrow \gamma,\ \mu\tilde d_n \Rightarrow \mu$, we arrive at the equation

$$z''+\sin(z)=\mu[(\cos(nz)+a)z'+\gamma] \tag{2.13}$$

According to [2], the generating function $\Phi(J)$ for (2.13) has the form

$$\Phi(J(\rho)) = \Phi^{(s)}(\rho) = aF_n^{(s)}(\rho)+F_0^s(\rho)\pm\delta_{2s}2\pi\gamma \tag{2.14}$$

$$F_0^{(1)}(\rho)=16[(\rho-1)K+E],\ F_1^{(1)}(\rho)=16[(1-\rho)K+(2\rho-1)E]/3$$

$$F_0^{(2)}(\rho)=8E/\sqrt{\rho},\ F_1^{(2)}(\rho)=8[2(\rho-1)K+(2-\rho)E]/3\rho^{3/2}$$

Here $s=1$ corresponds to the oscillatory domain and $s=2$ to the rotational domain, K and E are complete elliptic integrals, $\rho=k^2$, k is the module of elliptic integrals, $\rho=(1+\tilde{h})/2$ in the oscillatory domain and $\rho=2/(1+\tilde{h})$ in the rotational domain, $\tilde{h}=\tilde{h}(J(\rho))$ is the value of the energy integral of the equation $z''+\sin(z)=0$, $F_j^{(s)}(\rho)$ is the generating function determined by the perturbation term $z'\cos(jz)$, the plus sign corresponds to he upper half of the cylinder, the minus sign to the lower half, and δ_{zs} is the Kronecker delta.

Without assuming that μ is small, an equation of the form (2.13) with $n=1$ was considered in [10] in connection with the analysis of phase-locked automatic frequency control systems. Here, unlike [10], we use relationship (2.14) to study (2.13). This enables us to find all the bifurcation sets explicitly, except for one such set, namely, that corresponding to the saddle separatrix loop that fails to enclose the phase cylinder. We also remark that the results of [10] enable us to generalize the study of (2.13) to the case of Eq. (2.1) if $\sigma_* = dA/d\nu$.

We will first consider the case when $\gamma=0$. In this case $\Phi(\rho)$ is a continuous function when $\rho=1$. Then it determines the limit cycles up to the separatrix. This case was considered in [2]. In Fig. 2 we show the structurally stable topological structures for $n=1$. Note that limit cycles can diverge to "infinity" only when $B_1=0$. This is impossible when condition B is satisfied. In Fig. 3(a) we show the bifurcation case when the limit cycle is caught inside the separatrix contour (in this case $\Phi(\rho)$ has the simple root $\rho=1$). Figure 3(b) shows the corresponding behaviour of the invariant curves (separatrices) of the Poincaré map for the original system with $p=3$. The neighbourhood with homoclinic contour is attracting. Moreover, a non-trivial hyperbolic set exists in the neighbourhood itself [11] and, consequently, we have a "quasi-attractor" (chaotic dynamics–chaos).

When $\gamma\neq 0$ the generating function $\Phi(\rho)$ is discontinuous at $\rho=1$. Here the bifurcation of the cycle caught inside the separatrix must therefore be considered separately.

Using Mel'nikov's formula, we compute $\Delta_1^{\pm}$, which determines the splitting of the unperturbed separatrix for (2.13). One can see that the equation $\Delta_1^{\pm}=0$ is the same as $\Phi^{(2)}(1)=0$. Then, using (2.14) and assuming $n=1$ to be specific, we find the bifurcation values $\gamma^{\pm}=\mp 4(a+1/3)/\pi$. When $\gamma=\gamma^{+}+O(\mu)$ we have a saddle separatrix loop enclosing the phase cylinder and lying in the domain $z'>=0$, and when $\gamma=\gamma^{-}+O(\mu)$ – in the domain $z'<=0$. From (2.14) we obtain the following asymptotic forms for $\rho\Rightarrow 0$: $\Phi^{(2)}(\rho)\simeq\pi(8a/\sqrt{\rho}+\sqrt{\rho}\pm 4\gamma)/2$.

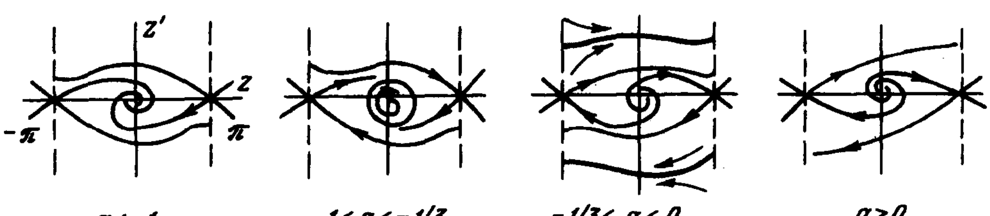

Fig. 2.

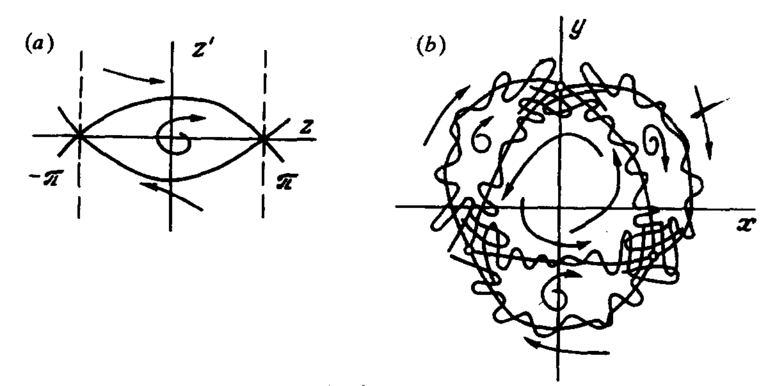

Fig. 3.

It follows that the straight line $a=0$ in the plane (a,γ) is singular. Furthermore, from (2.14) we find the dual cycle line in the parametric form

$$a=a_0(\rho)=-(F_0^{(2)})'/(F_n^{(2)})',\ \ \gamma=\gamma_0(\rho)=\mp(F_0^{(2)}(F_n^{(2)})'-$$
$$-(F_0^{(2)})'F_0^{(2)})/(2\pi\cdot(F_0^{(2)})'),\ \ \rho\in[0,1]\ \text{for}\ \ \gamma=\gamma_0^{\pm}(a)$$

It is seen that the modification of the phase portrait of Eq. (2.13) for $\rho \cong 1$ results in the formation of a saddle separatrix loop, which fails to enclose the phase cylinder. By condition B, $a\neq 0$, which implies that the saddle number is non-zero. Then the separatrix loop can give rise to only one limit cycle [9]. The corresponding bifurcation set $\gamma_1^{\pm}(a)$ in the parameter plane can be found numerically.

Hence we obtain a division of the parameter plane $(a,\ \gamma)$ into domains with different structures, as well as the topological structures for Eq. (2.13) themselves, which are shown in Fig. 4 for $n=1$. The structures corresponding to cases 8–12 are not shown in Fig. 4 because they can be obtained from structures 5, 6, 3, 2, and 14, respectively, by changing the directions of the coordinate axes.

Note that along with a separatrix loop enclosing the phase cylinder, Eq. (2.13) has a stable limit cycle that does or does not enclose the phase cylinder or, finally, a stable equilibrium state or stable "infinity". This means that no quasi-attractor can exist when $\gamma\neq 0$ in the original non-autonomous system. We remark that the homoclinic structure exists for a small range of variation of $\gamma(|\gamma-\gamma^{\pm}|\simeq\exp(-1/\mu))$.

We use (1.4) as an example in which a quasi-attractor exists when $\gamma\simeq 0$.

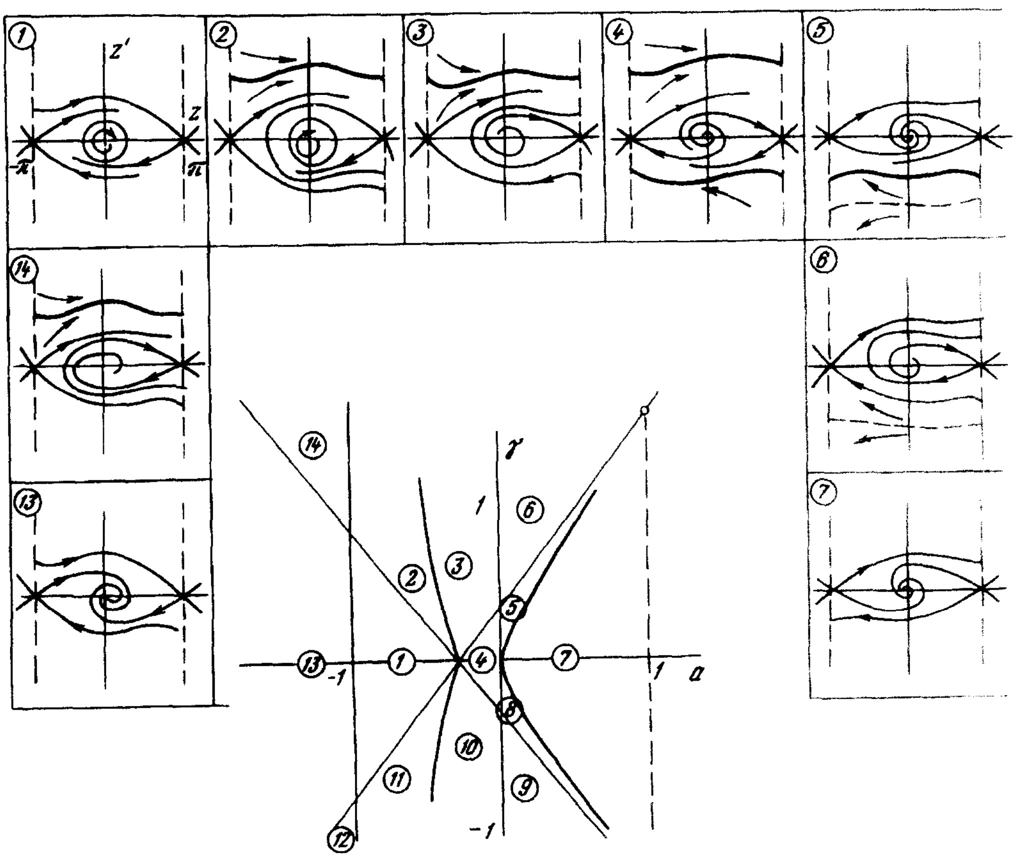

Fig. 4.

To those limit cycles of Eq. (2.13) that do not lie in the neighbourhood of the unperturbed separatrix contour there correspond two-dimensional invariant tori in the original system, as in the case $B \neq 0$. However, here, unlike the case $B \neq 0$, two kinds of torus may exist corresponding to the limit cycles in the oscillatory and rotational domains in (2.13). The tori corresponding to the cycles of Eq. (2.13) in the rotational domain (except for one of them) have no generating "Kolmogorov" torus in the corresponding perturbed Hamiltonian system (for system (1.4), $P_1 = P_2 = P_3 = 0$), while the asymptotically stable tori corresponding to the limit cycles of (2.13) in the oscillatory domain are images of the tori in the next step of the resonance hierarchy.

Remark. The cases when n is odd and even should be distinguished. When n is even, an unstable cycle is caught inside the separatrix loop. For odd n the cycle is stable. Only the case of odd n is therefore interesting when analysing the problem of the existence of a quasi-attractor.

According to the bifurcation diagram (Fig. 4), it is convenient to split the case $|a|<1$ into three subcases: (a) $-1<a<a_*$, δ), (b) $a_*<a<0$, and (c) $0<a<1$, $a_* = 1/(1-4n^2)$. On carrying over the results obtained for Eq. (2.13) to Eq. (2.12) with odd n, using [2] in the oscillatory domain, and considering the behaviour of solutions on the original cylinder $\{v(\mathrm{mod}\, 2\pi),\ u\}$, we arrive at the following theorem.

Theorem 2. There are μ_*, $\gamma^{\pm}(a)$, $\gamma_0^{\pm}(a)$, $\gamma_1^{\pm}(a)$, a_* such that if $|\mu|<\mu_*$ and n is odd, one can distinguish the following three intervals of a as in (2.12): (1) $a \in (-1,\ a_*)$, (2) $a \in (a_*,\ 0)$ and (3) $a \in (0,\ 1)$.

1. Let $a \in (-1,\ a_*)$. Then: (1) when $\gamma > \gamma_1^+ > 0$, Eq. (2.12) has a unique stable limit cycle (LC) enclosing the phase cylinder $\{v \,\mathrm{mod}\, 2\pi,\ u\}$ and no more than $p(n-1)$ LCs in the oscillatory domain (OD); (2) when $\gamma_1^+ < \gamma < \gamma^+$, there are p additional LCs in the OD, which are generated by the separatrix loops for $\gamma = \gamma_1^+$ (3) when $\gamma = \gamma^+$, a stable LC in the rotational domain is caught inside the separatrix contour Γ_p^+ consisting of p saddles and outer separatrices leading from one saddle to another, while the remaining unstable separatrices tend to an LC in the OD; (4) when $\gamma^- < \gamma < \gamma^+$, no LCs enclosing the phase cylinder exist and there are no more than pn LCs in the OD; (5) when $\gamma = \gamma^-$, a stationary contour Γ_p^- is formed consisting of p saddles and outer separatrices, which differs from Γ_p^+ by the direction in which the phase cylinder is circumvented and the location on that cylinder; (6) when $\gamma_1^- < \gamma < \gamma^-$, no more than $p \cdot n$ LCs exist in the OD and one stable LC enclosing the phase cylinder; (7) when $\gamma < \gamma_1^-$, Eq. (2.12) has a unique stable LC which encloses the phase cylinder $\{v \,\mathrm{mod}\, 2\pi,\ u\}$ and lies in the domain $u<0$, and no more than $p(n-1)$ LCs in the OD.
2. Let $a \in (a_*,\ 0)$. Then in the OD there are $p(n-1)$ LCs, and in the rotational domain: (1) when $\gamma > \gamma^-$, Eq. (2.12) has a unique stable LC for $u>0$; (2) when $\gamma = \gamma^-$, a contour Γ_p^- is formed; (3) when $\gamma^+ < \gamma < \gamma^-$, one stable LC exists on the upper half-cylinder ($u \geqslant 0$) and one stable LC on the lower half-cylinder ($u \leqslant 0$); (4) when $\gamma = \gamma^+$, a contour Γ_p^+ is formed; (5) when $\gamma < \gamma^+$, a unique stable LC exists for $u<0$.
3. Let $a \in (0,\ 1)$. Then there are no more than $p(n-1)$ LCs in the OD, and in the rotational domain: (1) when $\gamma > \gamma^-$ and $u<0$, Eq. (2.12) has a unique stable LC; (2) when $\gamma = \gamma^-$, a contour Γ_p is formed; (3) when $\gamma_0^- < \gamma < \gamma^-$ and $u<0$, there is a stable LC generated by Γ_p and an unstable LC; (4) when $\gamma = \gamma_0^-$, the stable and unstable LCs are merged together; (5) when $\gamma_0^+ < \gamma < \gamma_0^-$, no LCs exist; (6) when $\gamma = \gamma_0^+$ a semi-stable LC is formed for $u>0$ (7) when $\gamma^+ < \gamma < \gamma_0^+$, a stable and an unstable LC exist for $u>0$; (8) when $\gamma = \gamma^+$, the contour Γ_p^+ is formed; (9) when $\gamma < \gamma^+$, a unique unstable LC exists for $u>0$.

3. A COMPUTER ANALYSIS OF SYSTEM (1.4)

The present study of (1.4) supplements the earlier results [5]. We shall therefore only consider the effects due to the non-linear parametric term $xy\sin(vt)$. As in [5], we set $v=4$. Then for small P_i, $i=1$–4 system (1.4) can have only two "splittable" resonance levels: $H(x, y)=h_{11}$ and $H(x, y)=h_{31}$, where $h_{31}<h_{11}$. The corresponding autonomous system (1.3) has not more than one LC, the passage of which through the resonances when P_2 varies was considered in [5]. If the LC lies outside the neighbourhoods of the resonance levels $H(x, y)=h_{11}$ and $H(x, y)=h_{31}$, then there is a corresponding two-dimensional invariant torus T^2 in the original non-autonomous system (1.4) for which a generating "Kolmogorov" torus exists in the Hamiltonian system ($P_1=P_2=P_3=0$).

A program we developed was used in the computer analysis of (1.4). The results in Figs 5–8 were obtained using it. The numerical integration in this program involves Runge–Kutta type formulae, having an error of the order of $O(h^6)$ at each integration step h.

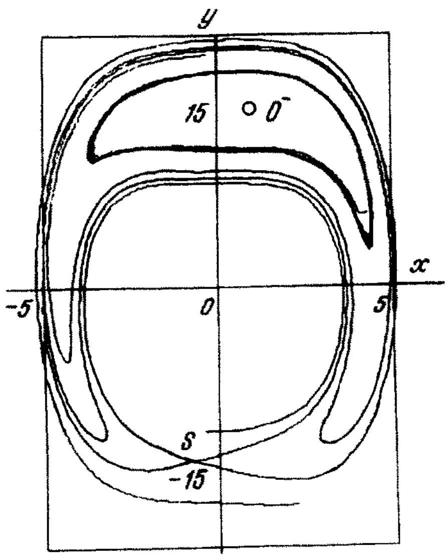

Fig. 5.

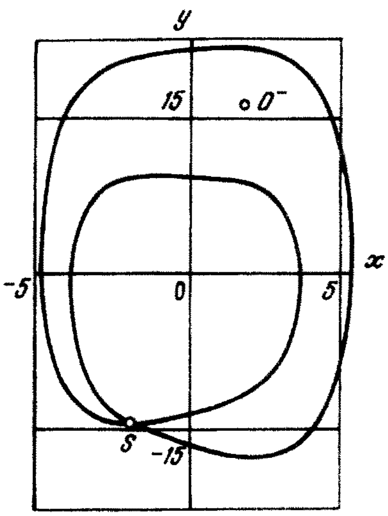

Fig. 6.

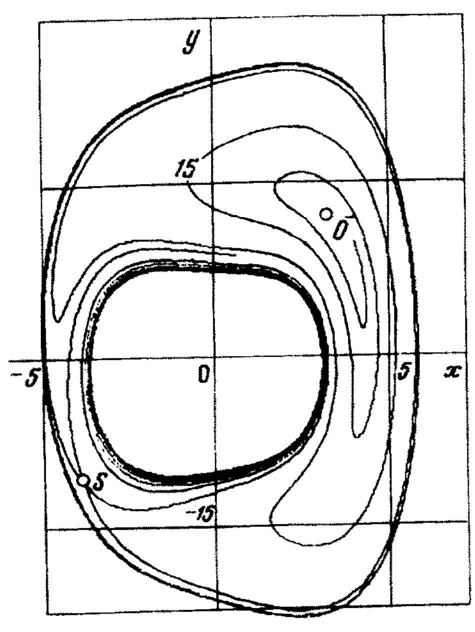

Fig. 7.

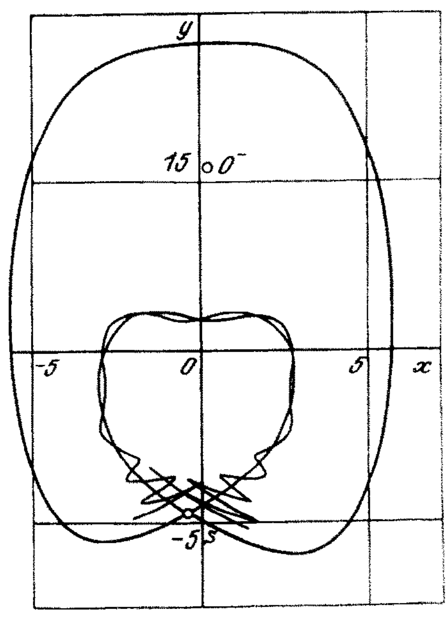

Fig. 8.

The numerical analysis of (1.4) was carried out for various values of P_4 (from 0.1 to 20) and for fairly small values of P_1, P_2, P_3. We fixed $P_2 = -0.008$ and varied P_2 and P_3 to study the fundamental resonance. It was established that the theoretical and numerical results are in good agreement as P_4 varies at least up to $P_4 = 2$. It is obvious that (1.4) cannot be considered as being close to an integrable system for $P_4 = 2$. From now on we assume everywhere that $P_4 = 2$ unless otherwise stipulated.

In Fig. 5 we present the Poincaré map for $P_1 = 0.0472$, $P_2 = -0.008$, and $P_3 = 0.018$, which determines the structure of the fundamental resonance zone ($p = 1$, $q = 1$). Along with the separatrices of the fixed saddle point S, a closed invariant curve enclosing the unstable fixed point O^- is shown, which corresponds to a stable LC in the oscillatory domain of (2.8). This closed invariant curve appears for $P_3 \approx 0.014$ as a result of the loss of stability of the fixed point O^-. As P_3 increases, the size of the closed invariant curve increases and for $P_3 \approx 0.0487$ the curve is caught inside the separatrix of the saddle point S, forming a contour (Fig. 6). As P_3 increases further two closed invariant curves appear, shown in Fig. 7 for $P_3 = 0.15$. These changes in the structure of the resonance zone are in good agreement with the theoretical results for $\gamma = 0$. Good agreement with the theory can also be seen for $\gamma \neq 0$.

In the case presented in Fig. 6 the transversality of the intersection of the separatrices of S cannot be detected visually. We therefore increased P_4 to obtain a better picture of the homoclinic structure. When $P_4 = 8$, the structure can be seen clearly (Fig. 8). The corresponding quasi-attractor is the only attracting set. Stable periodic points with long periods can exist inside the quasi-attractor itself. They are extremely difficult to detect numerically.

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